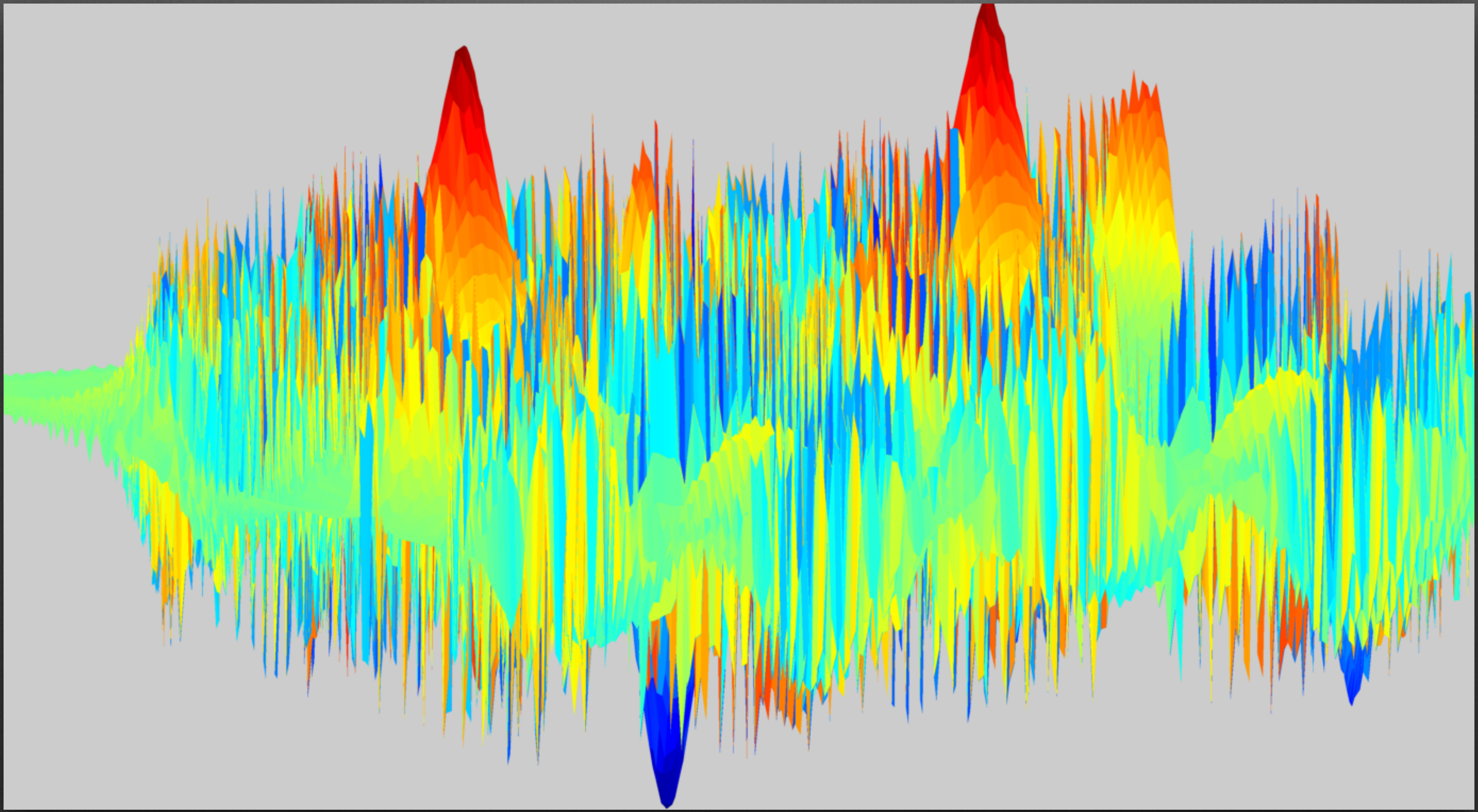
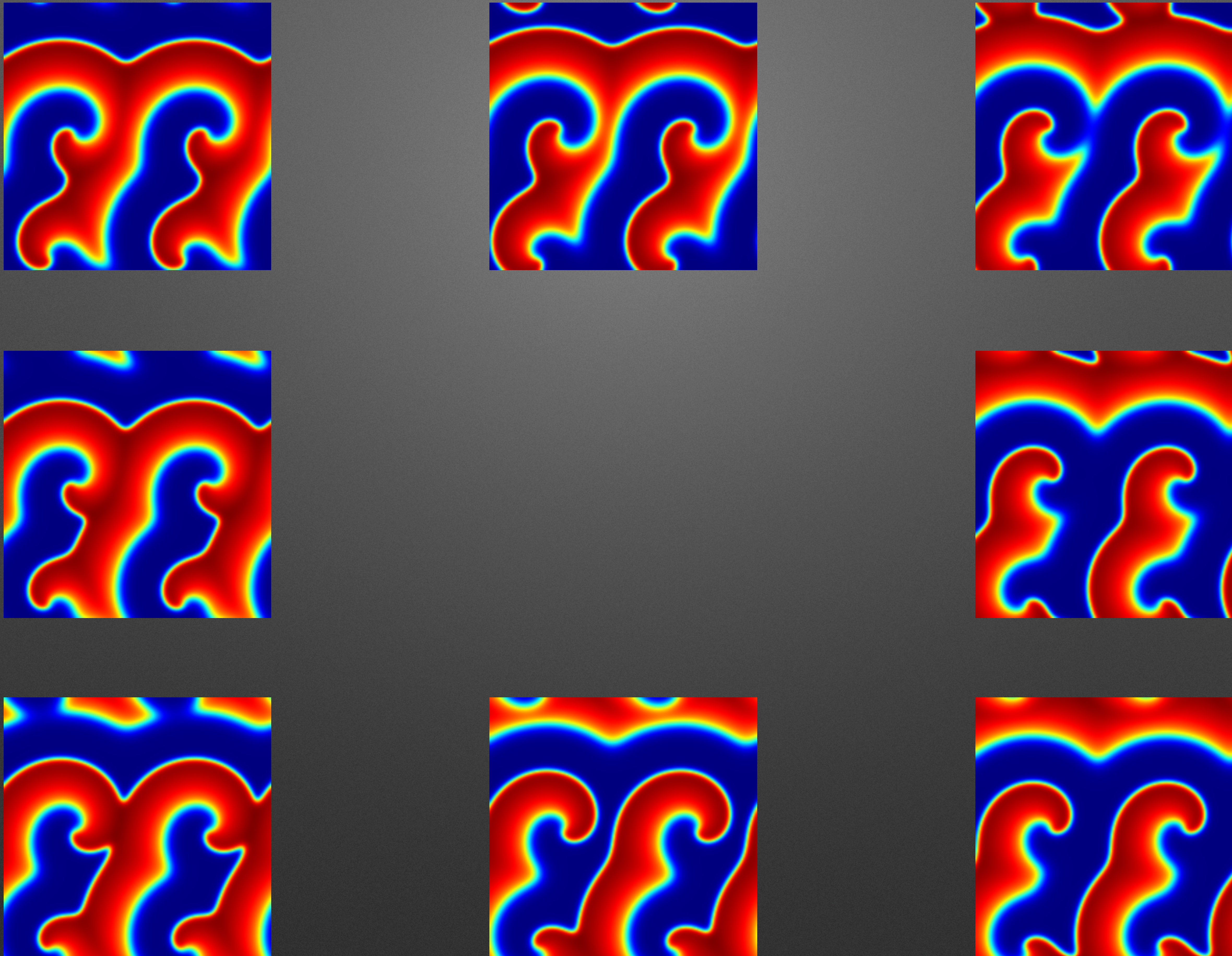


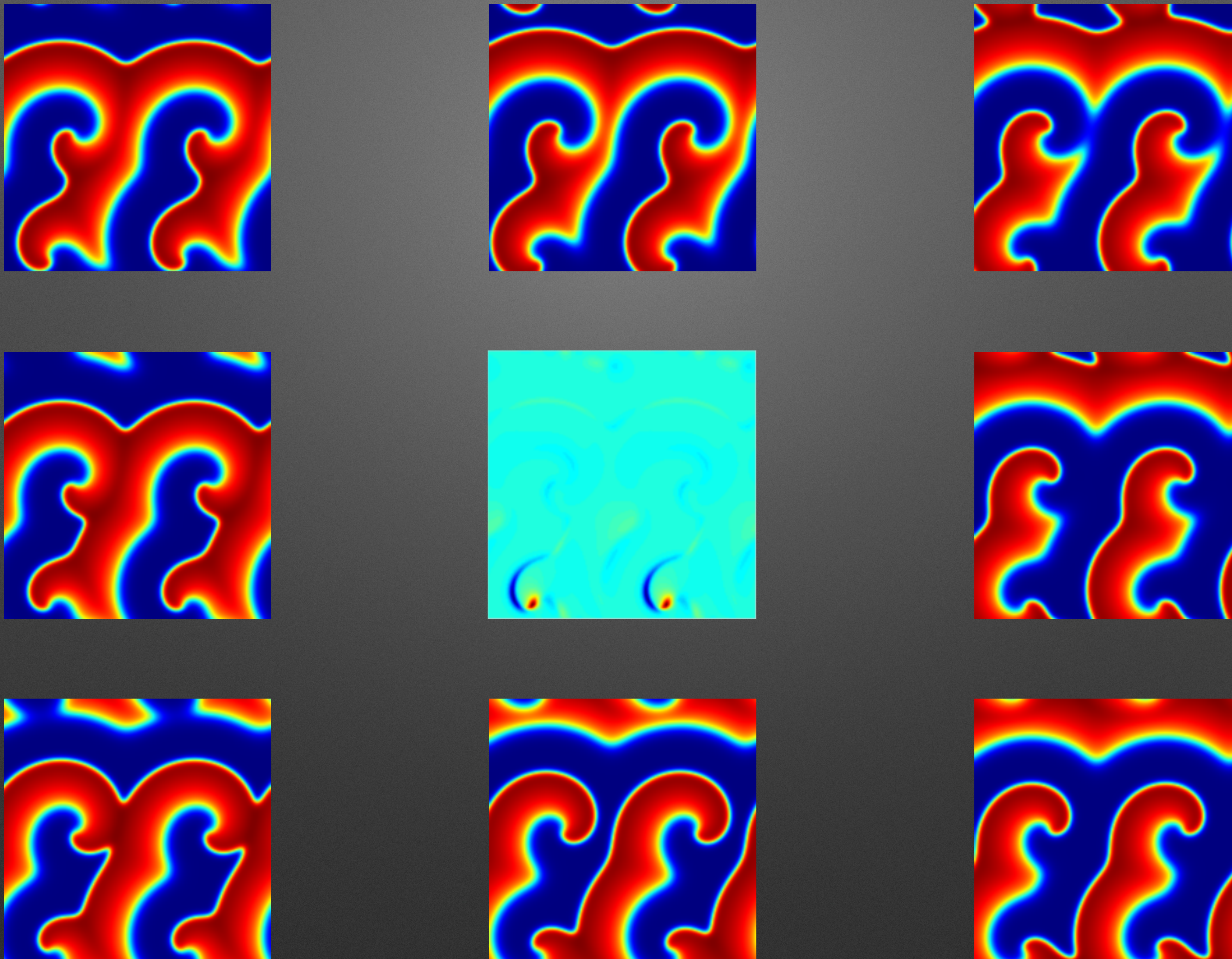
Core Exercises

Christopher Marcotte

Alice in the kingdom

- Separation of scales
    - fast dynamics
        - rotation, phase
    - slow dynamics
        - cores, amplitude
- Symmetries disorient GMRES
- Abundance of nearly closed trajectories
- *Why should every core drift the same way?*

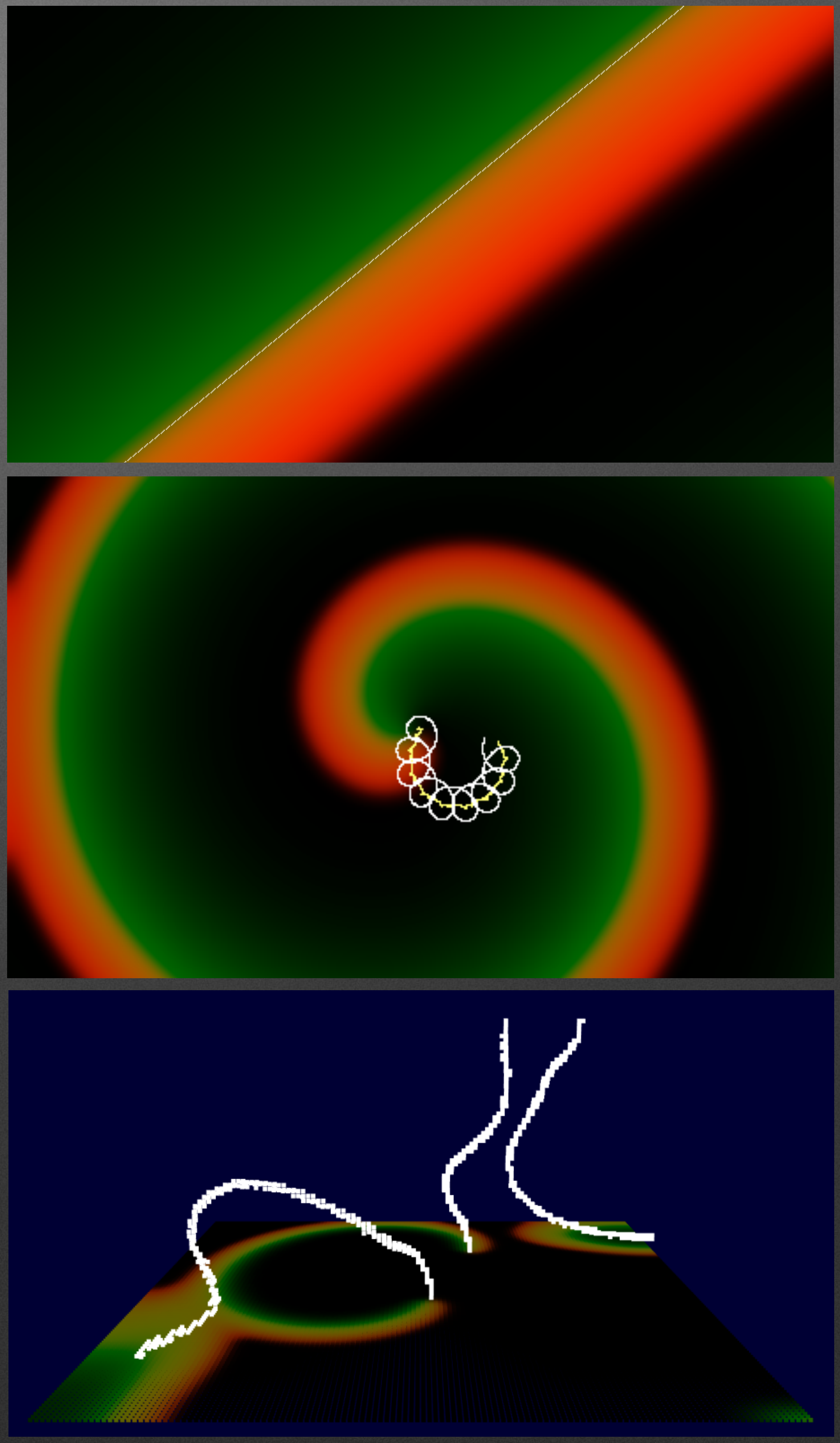




Reaction-Diffusion features

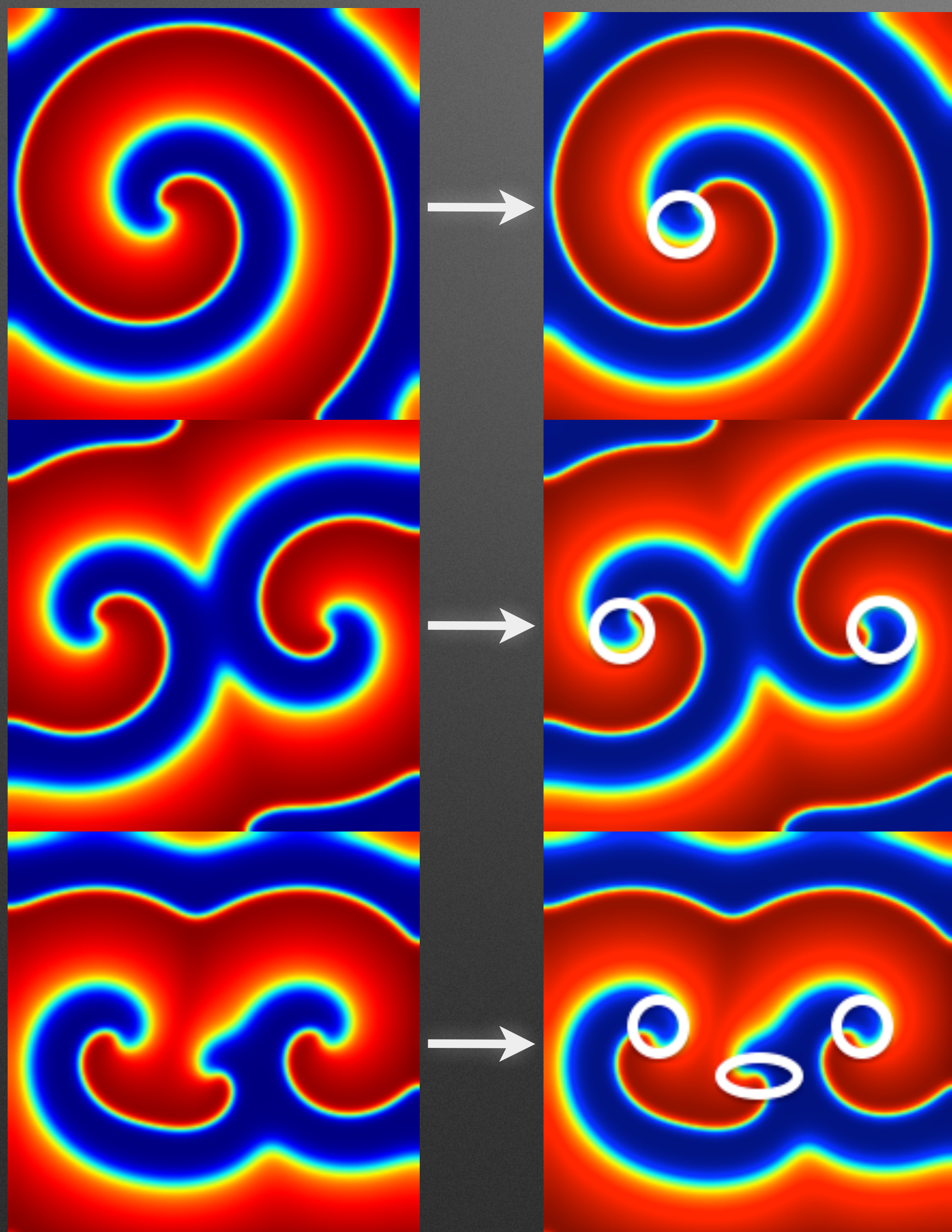
- 1-D systems have nodes
- 2-D systems have cores
- 3-D systems have filaments

- N-D have (N – 2)-D manifolds



ibiblio: Heart Rhythms

Off with their heads



- Cores are organizing features of the dynamics
- "Local" invariance inherited from global symmetries
- Tiling of domains of influence
- Reduce dynamics in each tile
- Tessellation of arbitrary domains by invariant solutions

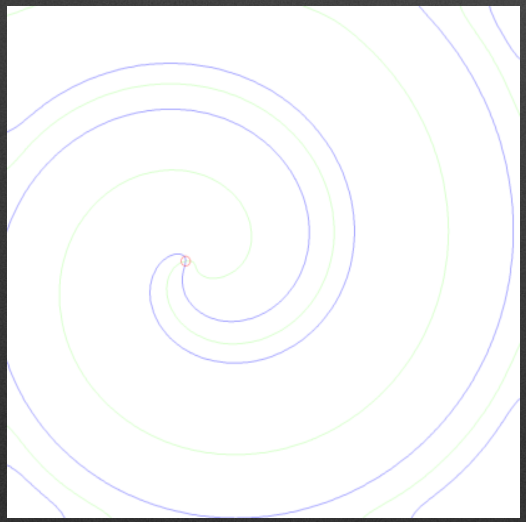
- Reduce the local symmetries around the important features of the flow
- Find invariants in the quotient space
- Construct families of symmetry related solutions with some set of reconstruction equations

First step is to find the cores

Two Paths Forward

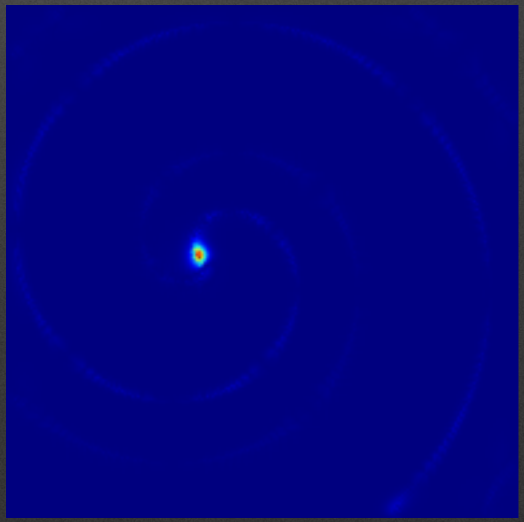
Level Sets

- Local
- Approximate
- Arbitrary



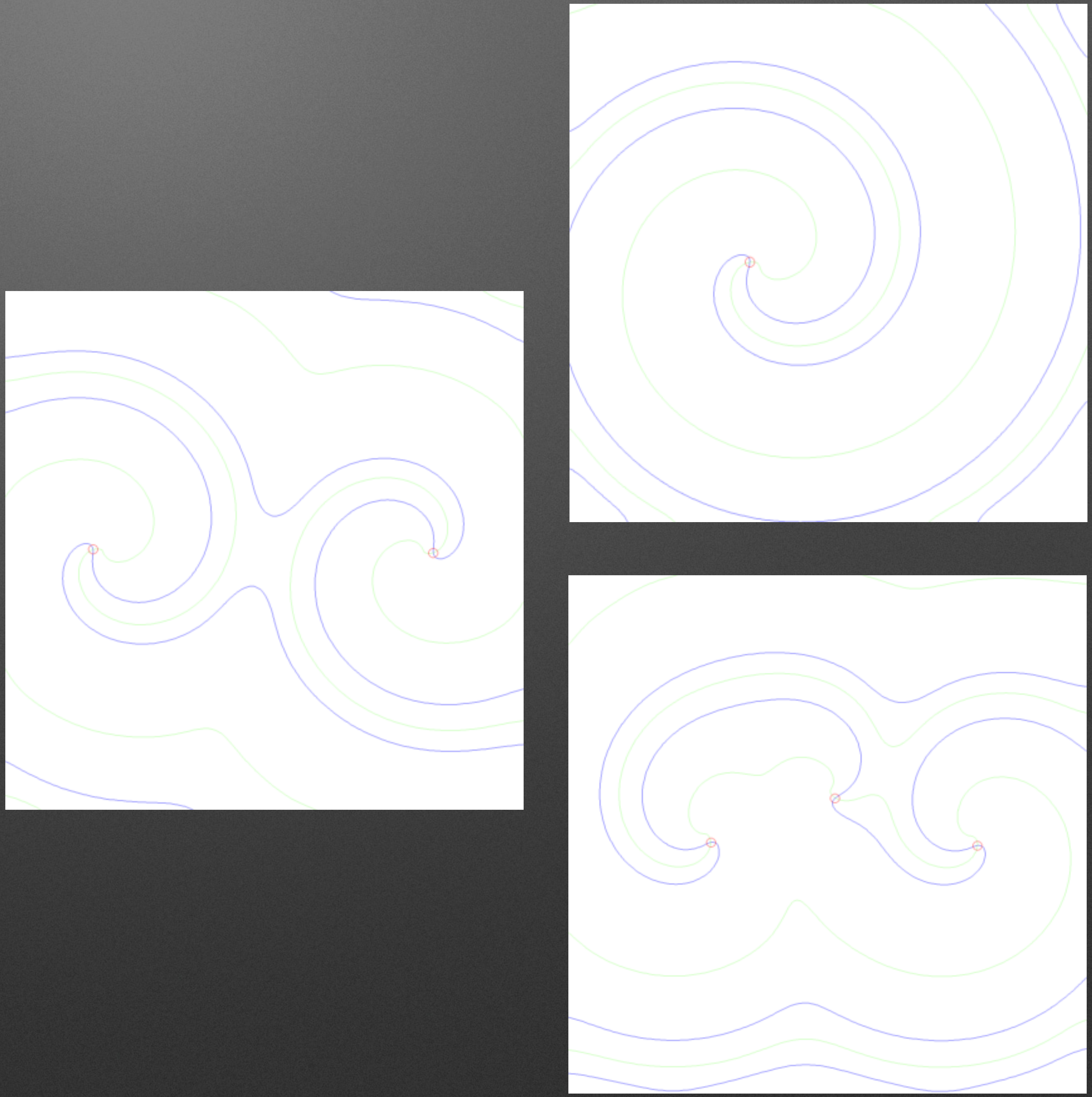
Singular Fields

- Global
- Iterative
- Topological



Level Sets

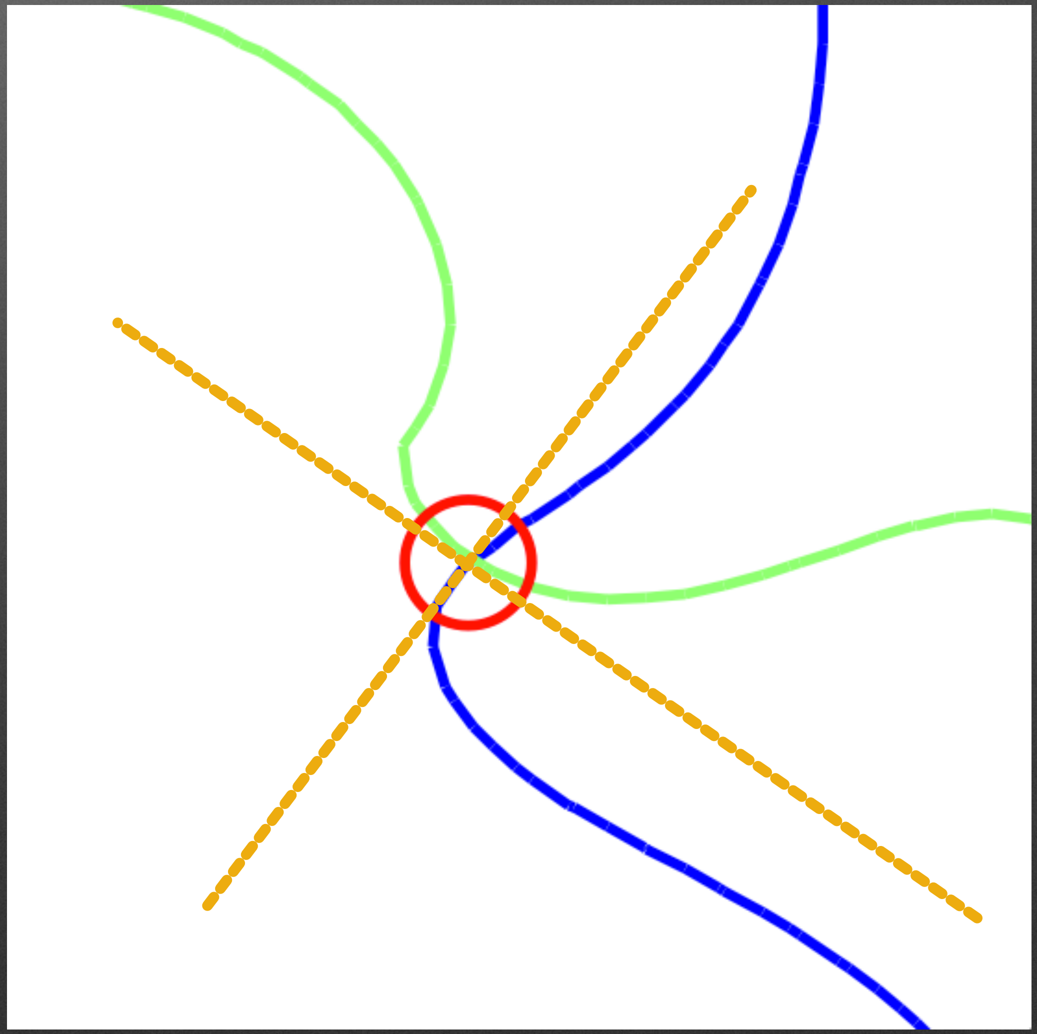
- Linear interpolation
- Simple, but limiting
- Arbitrary contour levels
- Fields, derivatives, etc.
  - The natural choice
  - The functional choice



Intersections

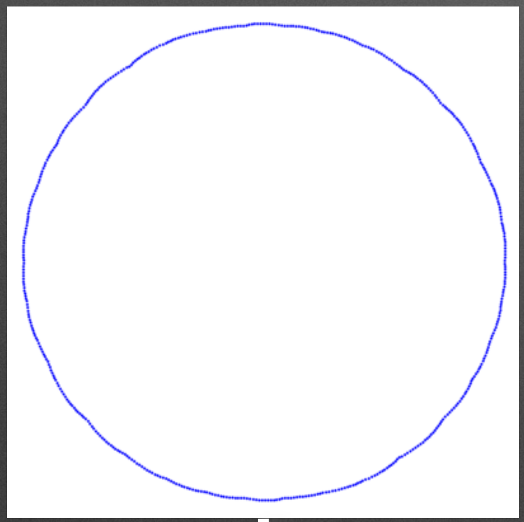
$$\{(x,y) : F[u(x,y)] = 0 \wedge G[v(x,y)] = 0\}$$

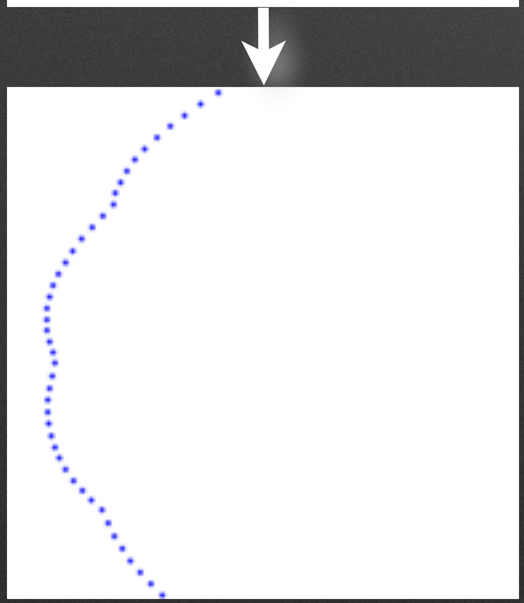
- Non-tangential crossings of level sets
- (Bi-)linear interpolation for sub-grid resolution
- Relatively smooth – no jumps
  - Non-smooth derivatives
- *Definitionally* 0-D
  - No ambiguity

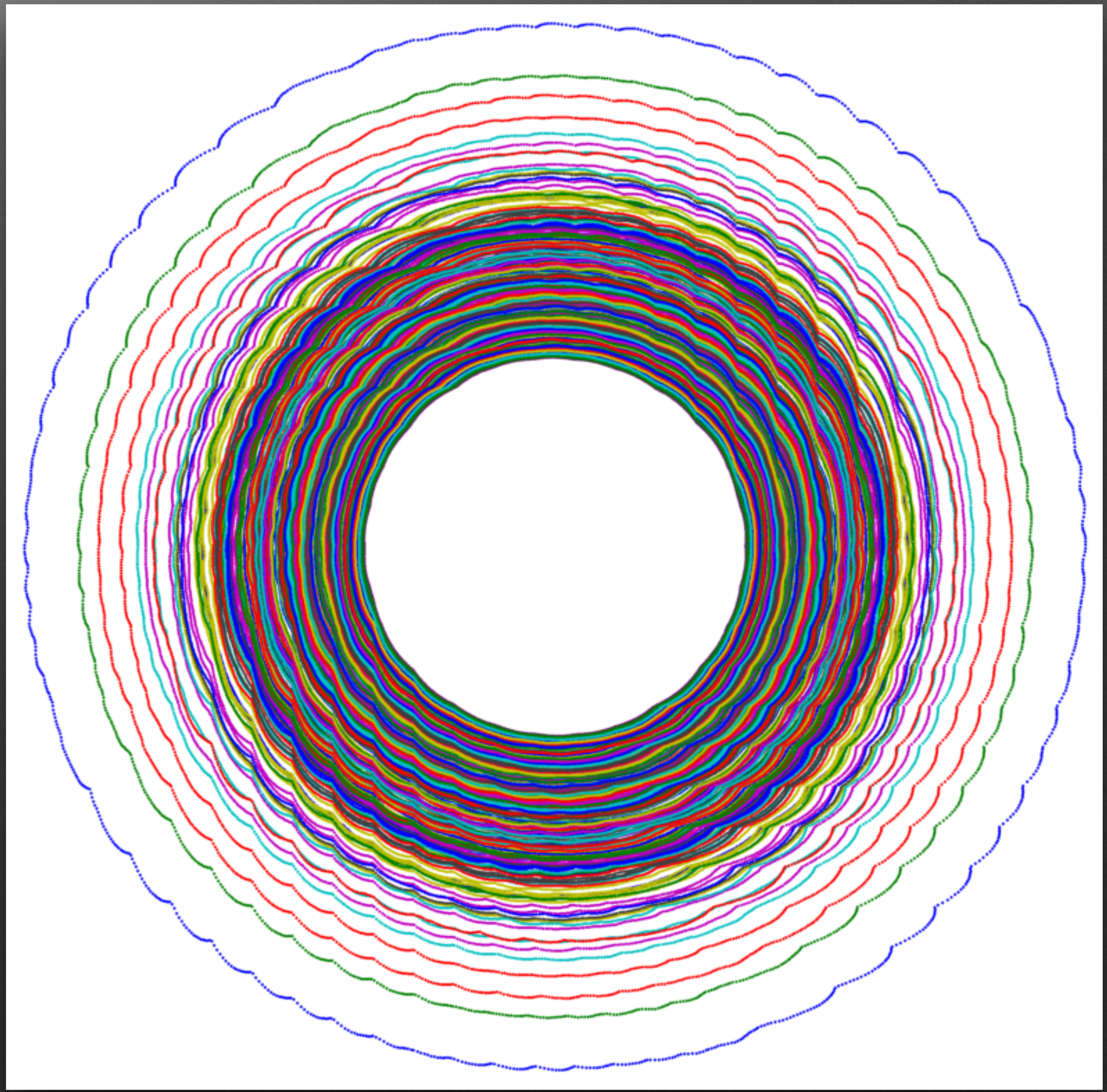


Interpolation

- Linear interpolation is discontinuous between grid points



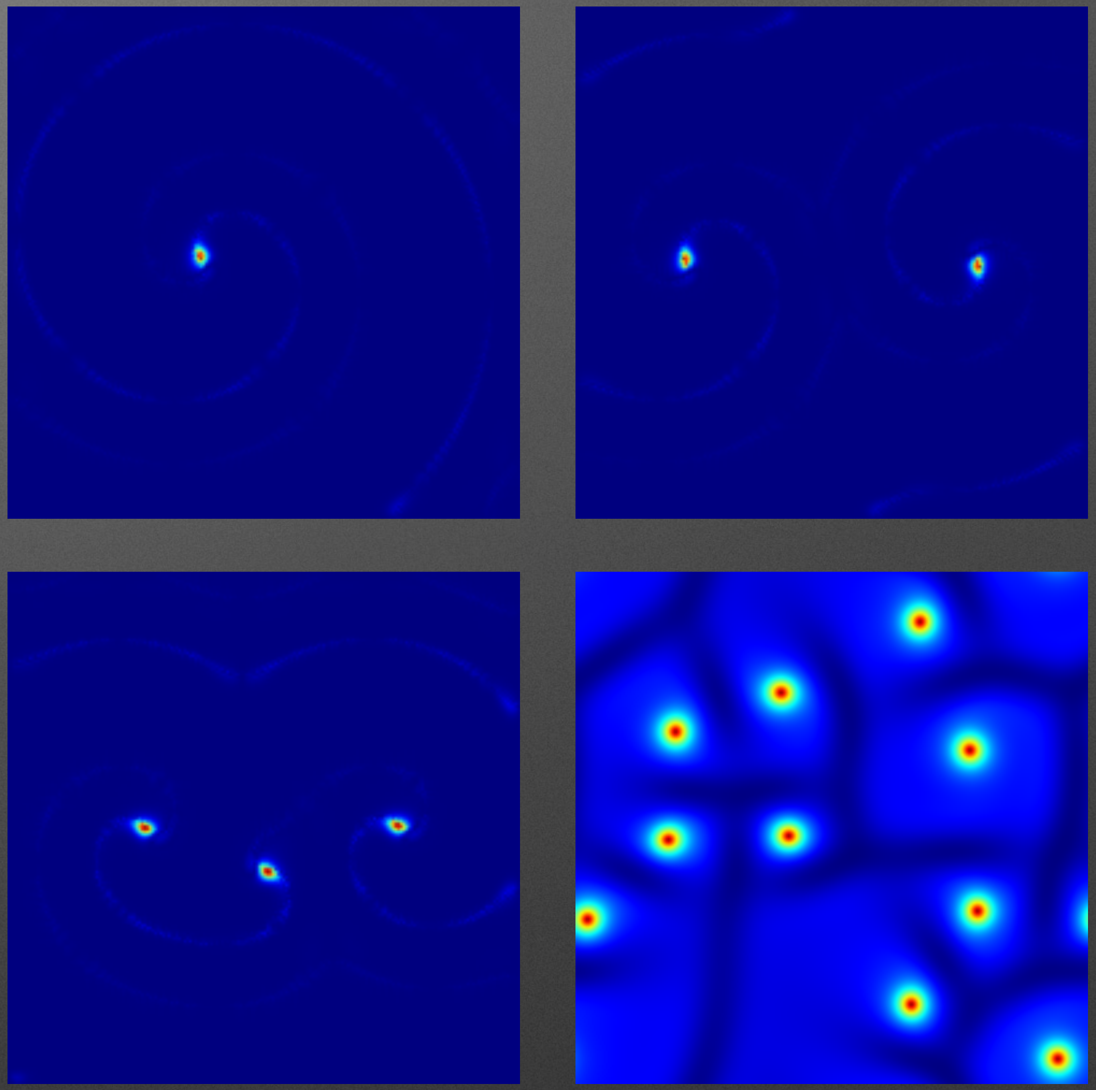




Minimizing core trajectory

Singular fields

- Definition:
  - Is unity 'at' the core, and zero elsewhere
- Amplitude *like* field
- Smoothness (!!)

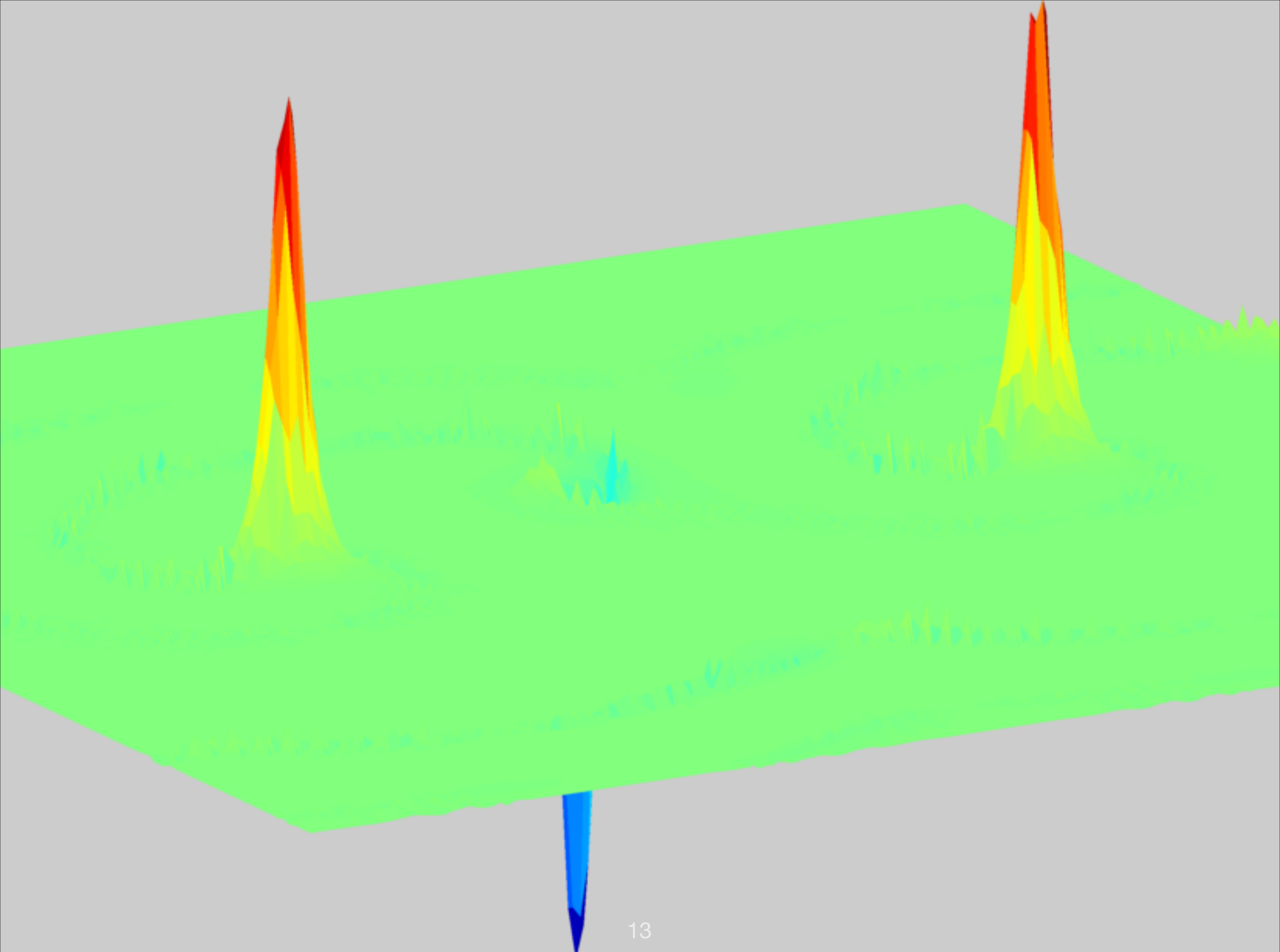


$$\omega_\times(\mathbf{x}) = \mathbf{e}_3 \cdot (\nabla u \times \nabla v)$$

$$\omega_\tau(\mathbf{x}) = (\varepsilon + \dot{u}^2 + \dot{v}^2)^{-1}$$

$$\omega_.(\mathbf{x}) = (\varepsilon + \nabla u \cdot \nabla v)^{-1}$$

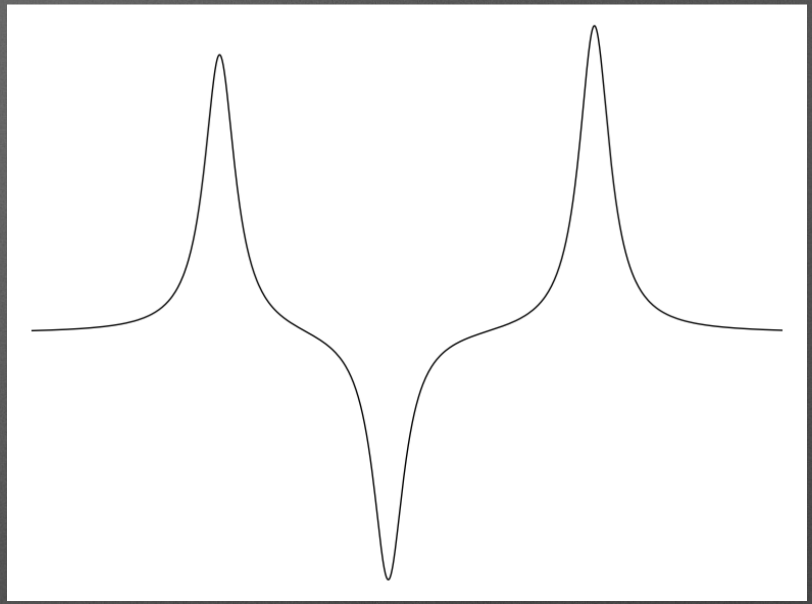
$$\omega_+(\mathbf{x}) = (\varepsilon + (u - \bar{u})^2 + (v - \bar{v})^2)^{-1}$$

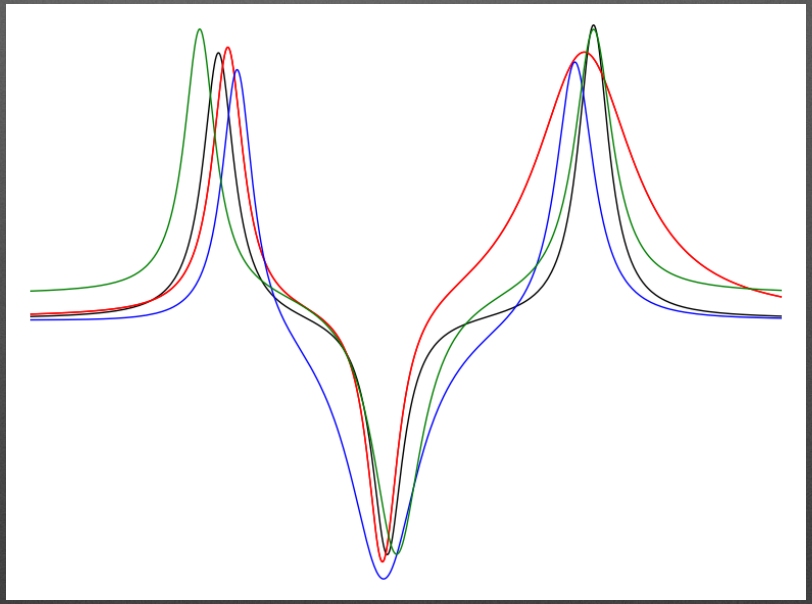


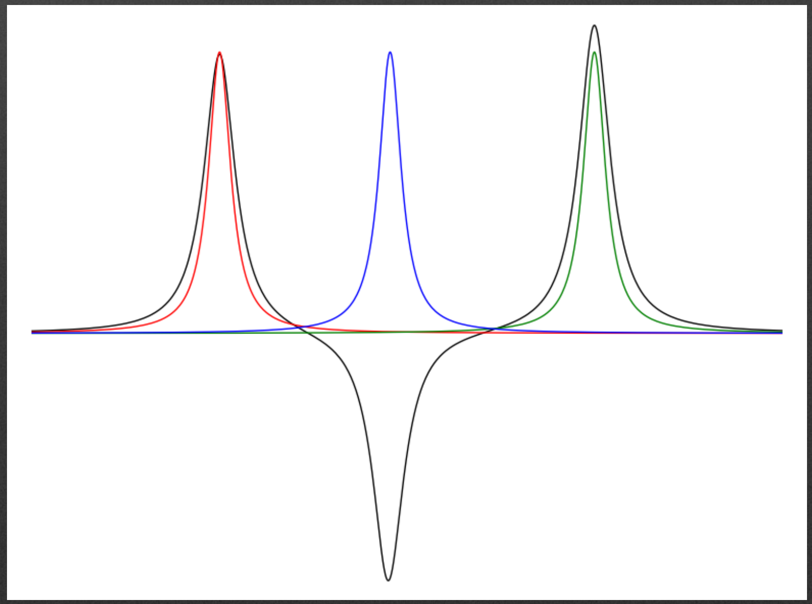
Minimization

$$\min_{\mathbf{x}'} \int_\Omega \mathrm{d}\mathbf{x}\, \ell(\mathbf{x} - \mathbf{x}')(1 - \omega[\mathbf{z}](\mathbf{x}))$$

- Global measure
- Integral formulation
- Iterative solution
  - Loop over cores
- Ansatz for core 'shape'
  - Lorentzian is effective
- Choice of singular field

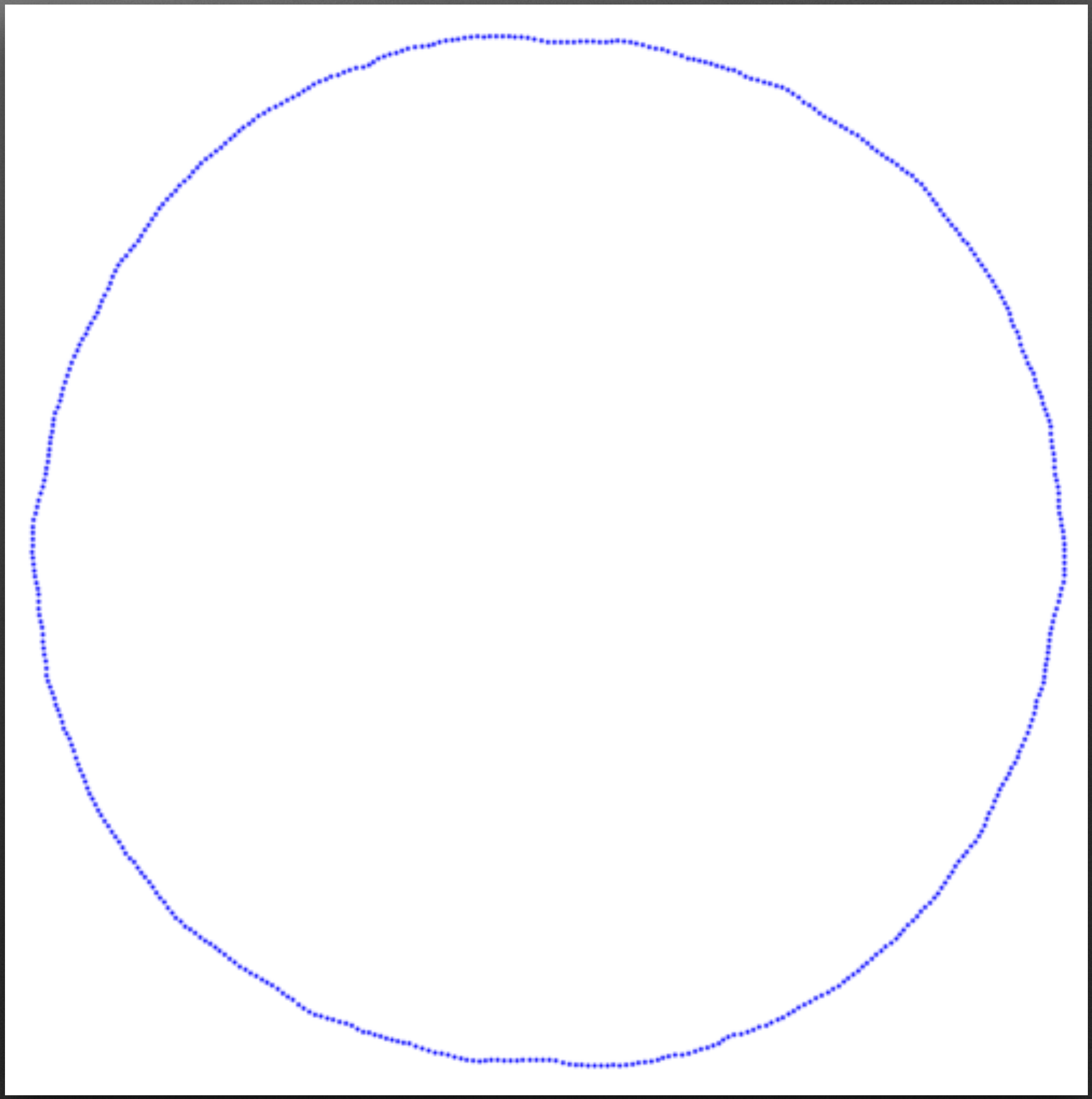






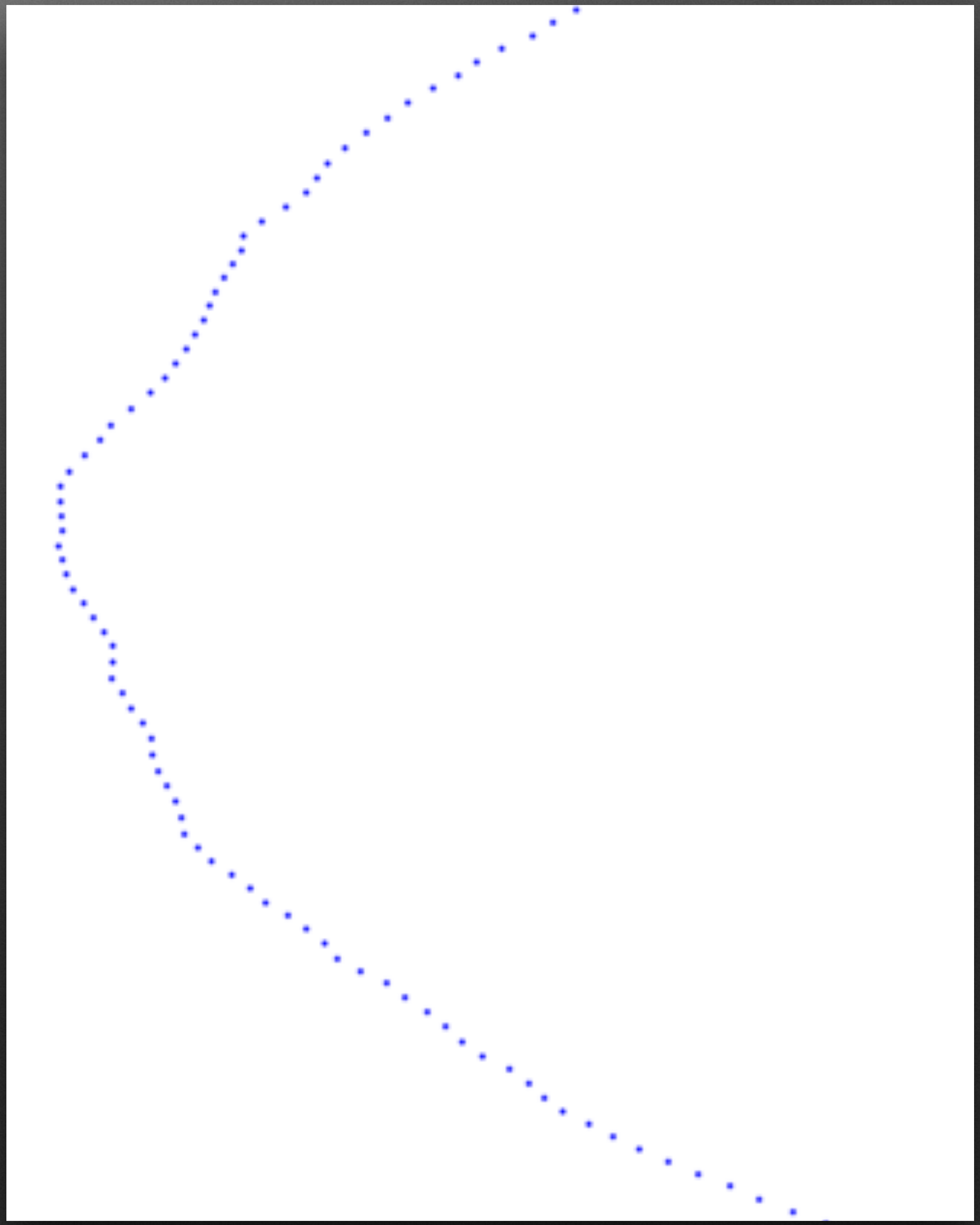
Minimization Technique

1. Define singular field
   - Which one?
2. Normalize appropriately
   - Field-dependent
3. Define fitting function
   - Smoothing / weighting
4. Apply iterative method
   - `fsolve` works fine



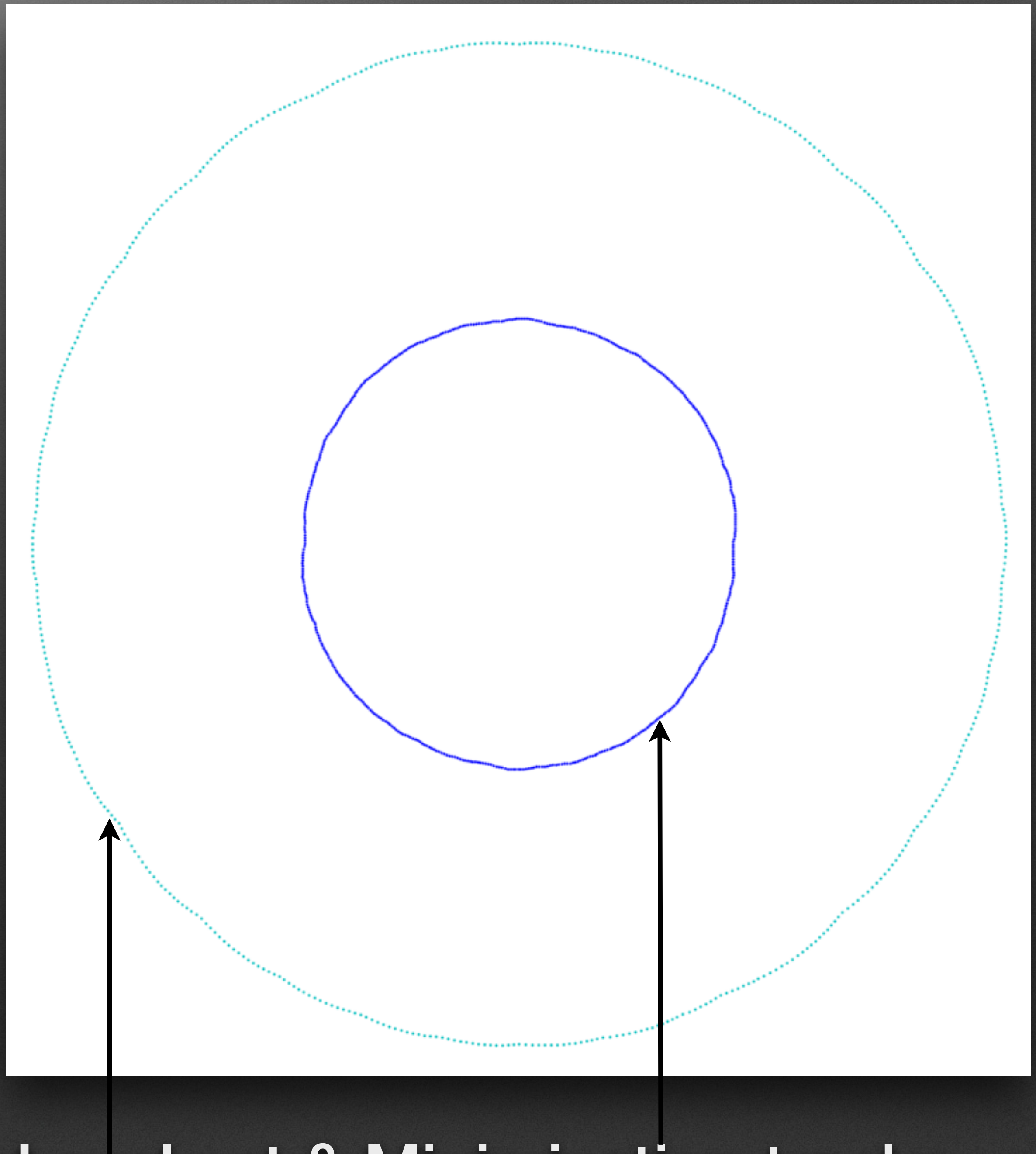
Limitations of Minimization

- Not continuous
  - Minimum can 'jump' due to non-smooth singularities
- Newton "trust region" search
  - Levenberg-Marquardt
- Weighting function sensitive
  - Lorentzian, Gaussian, etc.



Contours & Fields

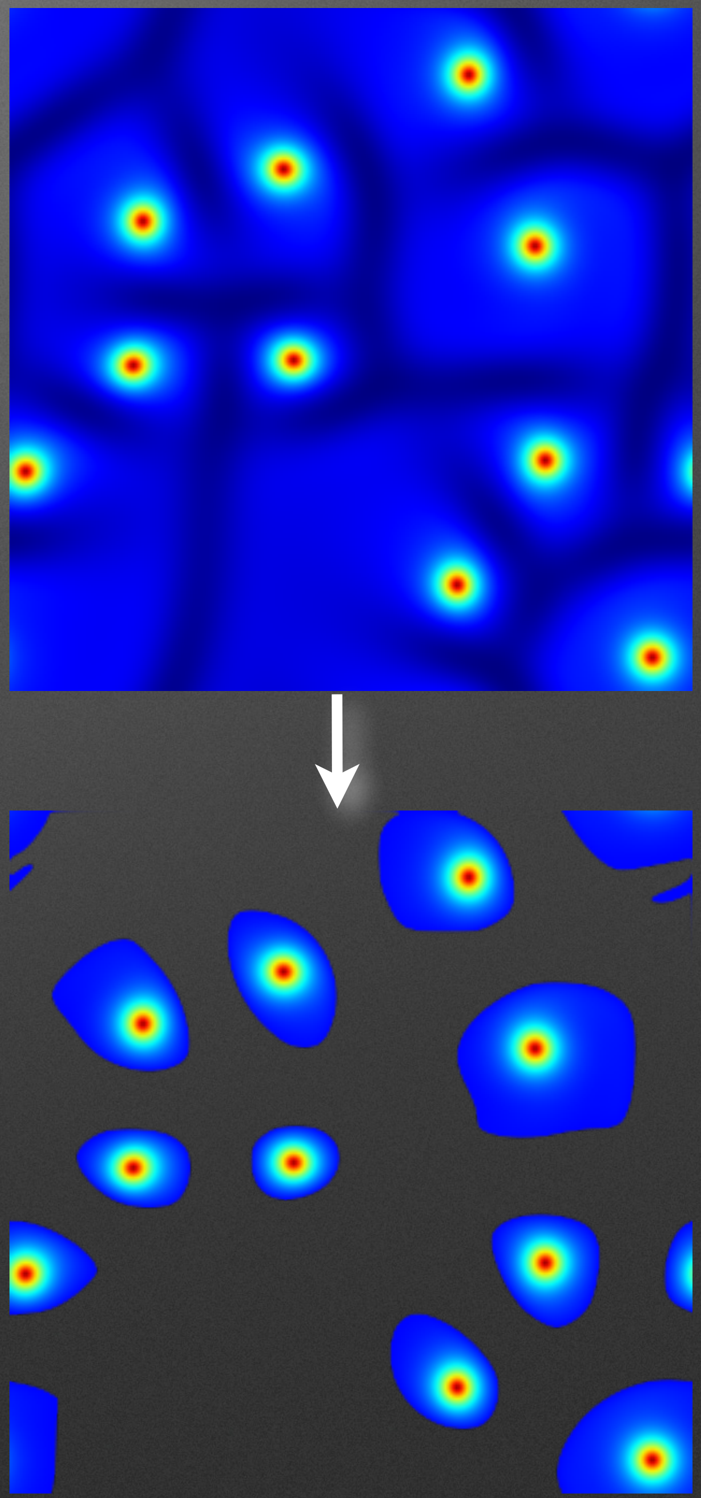
- Singular fields as a continuum analogue of the level sets
- Ironically, fields are less smooth on the sub-grid
- Natural choice for core, vanishing time-derivative, is least smooth of all
- Formulation which guarantees smoothness?
- Delay methods for cores?



Level set & Minimization tracks

Reconstruction - Tiling

- Compute entire set of global shifts for every tile
- Stitch global shifts together at tile borders
  - Potentially sensitive to stitching methods
- Interpretation as 'tree' of solutions in phase space
- Scaling is nearly optimal
  - ~ O(cores * symmetries)

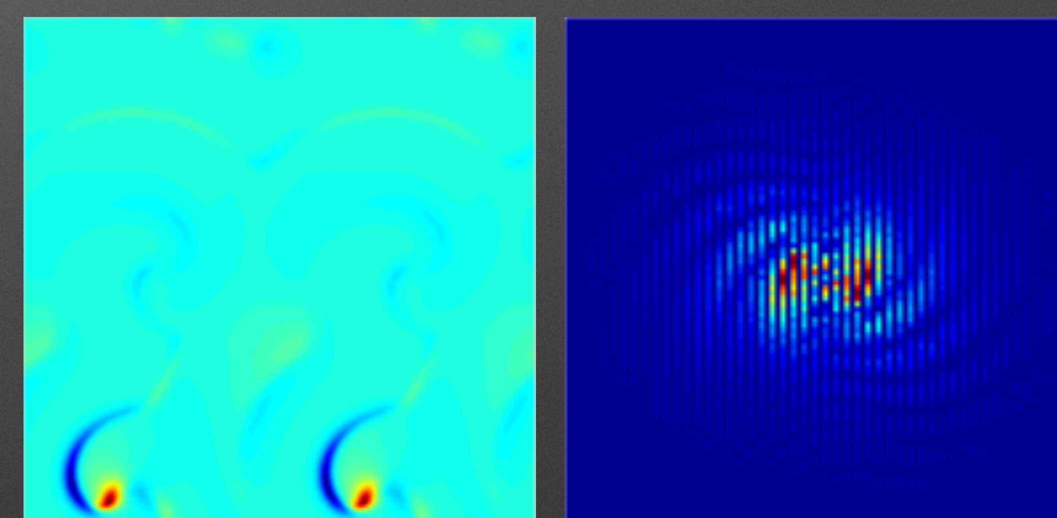


Reconstruction - Shearing

- Local dilation / contraction of solution
- Small corrections everywhere
    - spectral limit of tiling shifts
- Easy extension within GMRES
    - generalization of spatial shifts
    - *Now with more modes!*
        - How many modes?

$$\mathbf{z}' = \mathfrak{g}(\mathbf{s})\mathbf{z} = \left(\mathrm{e}^{-\mathbf{s}\cdot\nabla}\right)\mathbf{z}$$

$$\rightarrow \mathbf{z}' = \left(1 + \mathfrak{s}\cdot\nabla\right)\mathbf{z}$$



Residual and fft (zoomed)

Need small number of modes

Open Questions

- Optimal core definitions?
  - What is the source of non-smooth global tracking discontinuities?
  - How can we guarantee time-resolved core traces?
  - Is this method extendable?
- How do we reconstruct?
  - Tiling -- Does every 'stitched' solution work?
  - Shearing -- How do we truncate the number of modes?
- Interpretation of reconstructed solutions in state space?